

# IMPROVED LOWER BOUNDS FOR THE MAHLER MEASURE OF THE FEKETE POLYNOMIALS

TAMÁS ERDÉLYI

February 17, 2017

ABSTRACT. We show that there is an absolute constant  $c > 1/2$  such that the Mahler measure of the Fekete polynomials  $f_p$  of the form

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) z^k,$$

(where the coefficients are the usual Legendre symbols) is at least  $c\sqrt{p}$  for all sufficiently large primes  $p$ . This improves the lower bound  $(\frac{1}{2} - \varepsilon)\sqrt{p}$  known before for the Mahler measure of the Fekete polynomials  $f_p$  for all sufficiently large primes  $p \geq c_\varepsilon$ . Our approach is based on the study of the zeros of the Fekete polynomials on the unit circle.

## 1. INTRODUCTION AND NOTATION

Let  $D$  be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by  $\partial D$ . Let

$$\mathcal{K}_n := \left\{ P_n : P_n(z) = \sum_{k=0}^n a_k z^k, \ a_k \in \mathbb{C}, \ |a_k| = 1 \right\}.$$

The class  $\mathcal{K}_n$  is often called the collection of all (complex) unimodular polynomials of degree  $n$ . Let

$$\mathcal{L}_n := \left\{ P_n : P_n(z) = \sum_{k=0}^n a_k z^k, \ a_k \in \{-1, 1\} \right\}.$$

The class  $\mathcal{L}_n$  is often called the collection of all Littlewood polynomials of degree  $n$ . By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 dt = 2\pi(n+1)$$

---

*Key words and phrases.* polynomials, restricted coefficients, number of zeros on the unit circle, Legendre symbols, Fekete polynomials, Mahler measure.

*2010 Mathematics Subject Classifications.* 11C08, 41A17, 26C10, 30C15

for all  $P_n \in \mathcal{K}_n$ . Therefore

$$\min_{z \in \partial D} |P_n(z)| < \sqrt{n+1} < \max_{z \in \partial D} |P_n(z)|$$

for all  $P_n \in \mathcal{K}_n$  and  $n \geq 1$ . An old problem (or rather an old theme) is the following.

Let  $\alpha < \beta$  be real numbers. The Mahler measure  $M_0(P, [\alpha, \beta])$  is defined for bounded measurable functions  $P(e^{it})$  defined on  $[\alpha, \beta]$  as

$$M_0(P, [\alpha, \beta]) := \exp \left( \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \log |P(e^{it})| dt \right).$$

It is well known that

$$M_0(P, [\alpha, \beta]) = \lim_{q \rightarrow 0+} M_q(P, [\alpha, \beta]),$$

where, for  $q > 0$ ,

$$M_q(P, [\alpha, \beta]) := \left( \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} |P(e^{it})|^q dt \right)^{1/q}.$$

It is a simple consequence of the Jensen formula that

$$M_0(P) := M_0(P, [0, 2\pi]) = |c| \prod_{k=1}^n \max\{1, |z_k|\}$$

for every polynomial of the form

$$P(z) = c \prod_{k=1}^n (z - z_k), \quad c, z_k \in \mathbb{C}.$$

P. Borwein and Lockhart [B-01] investigated the asymptotic behavior of the mean value of normalized  $L_q$  norms of Littlewood polynomials for arbitrary  $q > 0$ . Using the Lindeberg Central Limit Theorem and dominated convergence, they proved that

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{(M_q(f, [0, 2\pi]))^q}{n^{q/2}} = \Gamma \left( 1 + \frac{q}{2} \right)$$

for every  $q > 0$ . In [C-15] we proved that

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{M_q(f, [0, 2\pi])}{n^{1/2}} = \left( \Gamma \left( 1 + \frac{q}{2} \right) \right)^{1/q}$$

for every  $q > 0$ . We also proved analogous results for the Mahler measure. Namely, using the notation  $\widehat{f}(z) := \max\{|f(z)|, n^{-1}\}$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \log \left( \frac{M_0(\widehat{f}, [0, 2\pi])}{n^{1/2}} \right) = -\gamma/2,$$

2

and

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{M_0(f, [0, 2\pi])}{n^{1/2}} = e^{-\gamma/2},$$

where

$$\gamma := \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \log n \right) = 0.577215 \dots$$

is the Euler constant and  $e^{-\gamma/2} = 0.749306 \dots$ . These are analogues of the results proved earlier by Choi and Mossinghoff [C-11] for polynomials in  $\mathcal{K}_n$ .

Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. Beller and Newman [B-73] constructed unimodular polynomials of degree  $n$  whose Mahler measure is at least  $\sqrt{n} - c/\log n$ . For a prime  $p$  the  $p$ -th Fekete polynomial is defined as

$$f_p(z) := \sum_{k=1}^{p-1} \left( \frac{k}{p} \right) z^k,$$

where

$$\left( \frac{k}{p} \right) = \begin{cases} 1, & \text{if } x^2 \equiv k \pmod{p} \text{ for an } x \neq 0, \\ 0, & \text{if } p \text{ divides } k, \\ -1, & \text{otherwise} \end{cases}$$

is the usual Legendre symbol. Note that  $g_p(z) := f_p(z)/z$  is a Littlewood polynomial, and has the same Mahler measure as  $f_p$ .

Montgomery [M-80] proved the following fundamental result.

**Theorem 1.1.** *There are absolute constants  $c_1 > 0$  and  $c_2 > 0$  such that*

$$c_1 \sqrt{p} \log \log p \leq \max_{z \in \partial D} |f_p(z)| \leq c_2 \sqrt{p} \log p.$$

In [E-07] we proved the following result.

**Theorem 1.2.** *For every  $\varepsilon > 0$  there is a constant  $c_\varepsilon$  such that*

$$M_0(f_p, [0, 2\pi]) \geq \left( \frac{1}{2} - \varepsilon \right) \sqrt{p}$$

for all primes  $p \geq c_\varepsilon$ .

From Jensen's inequality,

$$M_0(f_p) \leq M_2(f_p) = \sqrt{p-1}.$$

However, as it was observed in [E-07],  $\frac{1}{2} - \varepsilon$  in Theorem 1.2 cannot be replaced by  $1 - \varepsilon$ . Indeed, if  $p \geq 3$  is a prime and  $p = 4m + 1$ , then  $f_p$  is self-reciprocal, that is,  $z^p f_p(1/z) = f_p(z)$ , and hence

$$f_p(e^{2it}) = e^{ipt} \sum_{k=0}^{(p-3)/2} a_k \cos((2k+1)t), \quad a_k \in \{-2, 2\}.$$

Therefore a result of Littlewood [L-66] implies that

$$\begin{aligned} M_0(f_p) &\leq \frac{1}{2\pi} \int_0^{2\pi} |f_p(e^{iu})| du = \frac{1}{2\pi} \int_0^\pi |f_p(e^{2it})| 2dt = \frac{1}{2\pi} \int_0^{2\pi} |f_p(e^{2it})| dt \\ &\leq (1 - \varepsilon_0) \sqrt{p-1} \end{aligned}$$

with some absolute constant  $\varepsilon_0 > 0$ . If  $p \geq 3$  is a prime and  $p = 4m + 3$ , then  $f_p$  is anti-self-reciprocal, that is,  $z^p f_p(1/z) = -f_p(z)$ , and hence

$$if_p(e^{2it}) = e^{ipt} \sum_{k=0}^{(p-3)/2} a_k \sin((2k+1)t), \quad a_k \in \{-2, 2\}.$$

Therefore a result of Littlewood [L-66] implies that

$$\begin{aligned} M_0(f_p) &\leq \frac{1}{2\pi} \int_0^{2\pi} |if_p(e^{iu})| du = \frac{1}{2\pi} \int_0^\pi |if_p(e^{2it})| 2dt = \frac{1}{2\pi} \int_0^{2\pi} |if_p(e^{2it})| dt \\ &\leq (1 - \varepsilon_0) \sqrt{p-1} \end{aligned}$$

with some absolute constant  $\varepsilon_0 > 0$ .

It is an interesting open question whether there is a sequence of Littlewood polynomials  $(f_n)$  such that for an arbitrary  $\varepsilon > 0$ , and  $n$  large enough,

$$M_0(f_n) \geq (1 - \varepsilon) \sqrt{n}.$$

In [E-11] Theorem 1.2 was extended to subarcs of the unit circle.

**Theorem 1.3.** *There exists an absolute constant  $c_1 > 0$  such that*

$$M_0(f_p, [\alpha, \beta]) \geq c_1 p^{1/2}$$

*for all primes  $p$  and for all  $\alpha, \beta \in \mathbb{R}$  such that  $(\log p)^{3/2} p^{-1/2} \leq \beta - \alpha \leq 2\pi$ .*

In [E-12] we gave an upper bound for the average value of  $|f_p(z)|^q$  over any subarc  $I$  of the unit circle, valid for all sufficiently large primes  $p$  and all exponents  $q > 0$ .

**Theorem 1.4.** *There exists a constant  $c_2(q, \varepsilon)$  depending only on  $q > 0$  and  $\varepsilon > 0$  such that*

$$M_q(f_p, [\alpha, \beta]) \leq c_2(q, \varepsilon) p^{1/2},$$

*for all primes  $p$  and for all  $\alpha, \beta \in \mathbb{R}$  such that  $2p^{-1/2+\varepsilon} \leq \beta - \alpha \leq 2\pi$ .*

We remark that a combination of Theorems 1.3 and 1.4 shows that there is an absolute constant  $c_1 > 0$  and a constant  $c_2(q, \varepsilon) > 0$  depending only on  $q > 0$  and  $\varepsilon > 0$  such that

$$c_1 p^{1/2} \leq M_q(f_p, [\alpha, \beta]) \leq c_2(q, \varepsilon) p^{1/2}$$

for all primes  $p$  and for all  $\alpha, \beta \in \mathbb{R}$  such that  $(\log p)^{3/2} p^{-1/2} \leq 2p^{-1/2+\varepsilon} \leq \beta - \alpha \leq 2\pi$ .

The  $L_q$  norm of polynomials related to Fekete polynomials were studied in several recent papers. See [B-01b], [B-02], [B-04], [G-16], [J-13a], and [J-13b], for example. An interesting extremal property of the Fekete polynomials is proved in [B-01c].

Fekete might have been the first one to study analytic properties of the Fekete polynomials. He had an idea of proving non-existence of Siegel zeros (that is, real zeros “especially close to 1”) of Dirichlet  $L$ -functions from the positivity of Fekete polynomials on the interval  $(0, 1)$ , where the positivity of Fekete polynomials is often referred to as the Fekete Hypothesis. There were many mathematicians trying to understand the zeros of Fekete polynomials including Fekete and Pólya [F-12], Pólya [P-19], Chowla [C-35], Heilbronn [H-37], Montgomery [M-80], Baker and Montgomery [B-90], and Jung and Shen [J-16].

Baker and Montgomery [B-90] proved that  $f_p$  has a large number of zeros in  $(0, 1)$  for almost all primes  $p$ , that is, the number of zeros of  $f_p$  in  $(0, 1)$  tends to  $\infty$  as  $p$  tends to  $\infty$ , and it seems likely that there are, in fact, about  $\log \log p$  such zeros.

Conrey, Granville, Poonen, and Soundararajan [C-00] showed that  $f_p$  has asymptotically  $\kappa p$  zeros on the unit circle, where  $0.500668 < \kappa < 0.500813$ .

An interesting recent paper [B-17] studies power series approximations to Fekete polynomials.

It is conjectured, see [B-02] for instance, that there are sequences of flat Littlewood polynomials  $P_n \in \mathcal{L}_n$  satisfying

$$c_1 \sqrt{n+1} \leq |P_n(z)| \leq c_2 \sqrt{n+1}, \quad z \in \partial D,$$

with absolute constants  $c_1 > 0$  and  $c_2 > 0$ . However, the lower bound part of this conjecture, by itself, seems hard, and no sequence is known that satisfies just the lower bound. A sequence of Littlewood polynomials satisfying just the upper bound is given by the Rudin-Shapiro polynomials. They appear in Harold Shapiro’s 1951 thesis [S-51] at MIT and are sometimes called just Shapiro polynomials. They also arise independently in a paper by Golay (1951). They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures. The Rudin-Shapiro polynomials are defined recursively as follows:

$$\begin{aligned} P_0(z) &:= 1, & Q_0(z) &:= 1, \\ P_{n+1}(z) &:= P_n(z) + z^{2^n} Q_n(z), \\ Q_{n+1}(z) &:= P_n(z) - z^{2^n} Q_n(z), & n &= 0, 1, 2, \dots \end{aligned}$$

Note that both  $P_n$  and  $Q_n$  are polynomials of degree  $N - 1$  with  $N := 2^n$  having each of their coefficients in  $\{-1, 1\}$ . In [E-16] we showed that the Mahler measure and the maximum norm of the Rudin-Shapiro polynomials on the unit circle of the complex plane have the same size.

**Theorem 1.5.** *Let  $P_n$  and  $Q_n$  be the  $n$ -th Rudin-Shapiro polynomials defined in Section 1. There is an absolute constant  $c_1 > 0$  such that*

$$M_0(P_n, [0, 2\pi]) = M_0(Q_n, [0, 2\pi]) \geq c_1 \sqrt{N},$$

where

$$N := 2^n = \deg(P_n) + 1 = \deg(Q_n) + 1.$$

## 2. NEW RESULT

In this paper we improve the factor  $(\frac{1}{2} - \varepsilon)$  in Theorem 1.1 to an absolute constant  $c > 1/2$ . Namely we prove the following.

**Theorem 2.1.** *There is an absolute constant  $c > 1/2$  such that*

$$M_0(f_p) \geq c\sqrt{p}$$

for all sufficiently large primes.

## 3. LEMMAS

To prove the theorem we need a few lemmas. For a natural number  $p$  let

$$\zeta_p := \exp\left(\frac{2\pi i}{p}\right)$$

be the first  $p$ -th root of unity. Our first lemma formulates an characteristic property of the Fekete polynomials. A simple proof is given in [B-02, pp. 37-38].

**Lemma 3.1 (Gauss).** *We have*

$$f_p(\zeta_p^j) = \sqrt{\left(\frac{-1}{p}\right)p}, \quad j = 1, 2, \dots, p-1,$$

and  $f_p(1) = 0$ .

**Lemma 3.2.** *We have*

$$\left(\prod_{j=0}^{p-1} |Q(\zeta_p^j)|\right)^{1/p} \leq 2M_0(Q)$$

for all polynomials  $Q$  of degree at most  $p$  with complex coefficients.

**Lemma 3.3.** *Let  $0 < \eta \leq \pi/2$  be fixed. Suppose a polynomial  $Q$  of degree at most  $p$  with complex coefficients has at least  $k$  zeros*

$$b_j = e^{it_j}, \quad j = 1, 2, \dots, k,$$

*such that*

$$t_j \in [0, 2\pi) \setminus \bigcup_{\nu=0}^{p-1} \left( \frac{(2\nu+1)\pi}{p} - \frac{\eta}{p}, \frac{(2\nu+1)\pi}{p} + \frac{\eta}{p} \right).$$

*We have*

$$\left( \prod_{j=0}^{p-1} |Q(\zeta_p^j)| \right)^{1/p} \leq 2 \left( \cos \frac{\eta}{2} \right)^{k/p} M_0(Q).$$

In the proof of Theorem 2.1 we need one of the following two results. For proofs see [B-97a] and [B97-b], respectively.

**Lemma 3.4.** *There is an absolute constant  $c > 0$  such that every  $Q \in \mathcal{K}_n$  has at most  $c\sqrt{n}$  real zeros.*

**Lemma 3.5.** *There is an absolute constant  $c > 0$  such that every  $Q \in \mathcal{L}_n$  has at most  $\frac{c \log^2 n}{\log \log n}$  zeros at 1.*

The large sieve of number theory [M-78] asserts the following.

**Lemma 3.6.** *If*

$$P(z) = \sum_{k=-n}^n a_k z^k, \quad a_k \in \mathbb{C},$$

*is a trigonometric polynomial of degree at most  $n$ ,*

$$0 \leq t_1 < t_2 < \dots < t_m \leq 2\pi,$$

*and*

$$\delta := \min \{t_2 - t_1, t_3 - t_2, \dots, t_m - t_{m-1}, 2\pi - (t_m - t_1)\},$$

*then*

$$\sum_{j=1}^m |P(e^{it_j})|^2 \leq \left( \frac{2n+1}{2\pi} + \delta^{-1} \right) \int_0^{2\pi} |P(e^{it})|^2 dt.$$

It turns out to be fairly easy to show that at least half of the zeros of  $f_p$  are on the unit circle  $\partial D$ . First note that

$$F_p(z) := z^{-p/2} f_p(z) = \sum_{a=1}^{(p-1)/2} \left( \frac{a}{p} \right) \left( z^{a-p/2} + \left( \frac{-1}{p} \right) z^{p/2-a} \right).$$

Observe also that

$$(3.1) \quad F_p(e^{2i\pi t}) := \begin{cases} 2 \sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) \cos((2a-p)\pi t) & \text{if } p \equiv 1 \pmod{4} \\ 2i \sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) \sin((2a-p)\pi t) & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Define  $H_p(t) := F_p(e^{2i\pi t})$  if  $p \equiv 1 \pmod{4}$ , and  $H_p(t) := -iF_p(e^{2i\pi t})$  if  $p \equiv 3 \pmod{4}$ . By (3.1) we see that  $H_p(t)$  is a periodic, continuous, real-valued function when  $t$  is real.

**Lemma 3.7.** *Let  $p$  be a prime. There are at least  $(p-3)/2$  values of  $k \in \{0, 1, \dots, p-1\}$  for which  $H_p$  has a zero between  $k/p$  and  $(k+1)/p$ .*

Our next lemma is Theorem 4 in [C-00]. For a proof of Lemma 3.8 below see Section 6 in [C-00].

**Lemma 3.8.** *Let  $p$  be a prime. For every fixed real number  $\delta$*

$$\left| \left\{ k \in \{1, 2, \dots, p\} : H_p\left(\frac{k+1/2}{p}\right) < \delta\sqrt{p} \right\} \right| \sim c_\delta p$$

as  $p \rightarrow \infty$ , where

$$c_\delta = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \sin(\delta\pi x) C(x) \frac{dx}{x}, \quad C(x) := \prod_{k=0}^\infty \cos^2\left(\frac{2x}{2k+1}\right).$$

Moreover  $c_{-\delta} = 1 - c_\delta$  for all  $\delta > 0$ .

**Lemma 3.9.** *For every  $\varepsilon > 0$  there is a  $\delta > 0$  such that*

$$\left| \left\{ k \in \{1, 2, \dots, p\} : H_p\left(\frac{k+1/2}{p}\right) \geq \delta\sqrt{p} \right\} \right| \geq (1 - \varepsilon)p$$

for all sufficiently large primes  $p \geq N_\varepsilon$ .

**Lemma 3.10.** *Let  $\gamma > 0$  be a real number. Let the subarcs  $I_k$  of the unit circle  $\partial D$  be defined by*

$$I_k := \left\{ e^{it} : \left| t - \frac{(2k+1)\pi}{p} \right| \leq \frac{\pi}{2p} \right\}, \quad k = 0, 1, \dots, p-1.$$

We have

$$m := |\{k \in \{0, 1, \dots, p-1\} : \max_{z \in I_k} |f'_p(z)| \geq \gamma p^{3/2}\}| \leq \gamma^{-2} p/2$$

for all primes  $p \geq 3$ .

**Lemma 3.11.** *Given  $\eta > 0$  let the subarcs  $I_{k,\eta}$  of the unit circle  $\partial D$  be defined by*

$$I_{k,\eta} := \left\{ e^{it} : \left| t - \frac{(2k+1)\pi}{p} \right| < \frac{\eta}{p} \right\}, \quad k = 0, 1, \dots, p-1.$$

For every  $\varepsilon > 0$  there is an  $\eta > 0$  such that

$$|\{k \in \{0, 1, \dots, p-1\} : f_p(z) \neq 0 \text{ for all } z \in I_{k,\eta}\}| \geq (1 - \varepsilon)p$$

for all sufficiently large primes  $p \geq N_\varepsilon$ .



#### 4. PROOFS OF THE LEMMAS

*Proof of Lemma 3.2.* Let

$$Q(z) = c \prod_{j=1}^m (z - a_j), \quad c, a_j \in \mathbb{C},$$

with some  $m \leq p$ . Without loss of generality we may assume that  $c = 1$ . Note that

$$|a_j^p - 1|^{1/p} \leq (2|a_j|^p)^{1/p} = 2^{1/p}|a_j|, \quad |a_j| \geq 1,$$

while

$$|a_j^p - 1|^{1/p} \leq 2^{1/p}, \quad |a_j| < 1.$$

Multiplying these inequalities for  $j = 1, 2, \dots, m$ , we obtain

$$\left( \prod_{j=0}^{p-1} |Q(\zeta_p^j)| \right)^{1/p} = \left( \prod_{j=0}^m |a_j^p - 1| \right)^{1/p} \leq 2^{m/p} \prod_{j=0}^m \max\{|a_j|, 1\} \leq 2M_0(Q).$$

□

*Proof of Lemma 3.3.* Let

$$Q(z) = c \prod_{j=1}^m (z - a_j), \quad c, a_j \in \mathbb{C},$$

with some  $m \leq p$ , where  $a_j = b_j$ ,  $j = 1, 2, \dots, k$ . Without loss of generality we may assume that  $c = 1$ . Note that

$$|a_j^p - 1|^{1/p} \leq 2^{1/p} \max\{|a_j|, 1\}, \quad j = k+1, k+2, \dots, m,$$

and

$$|a_j^p - 1|^{1/p} \leq \left( 2 \cos \frac{\eta}{2} \right)^{1/p} = \left( 2 \cos \frac{\eta}{2} \right)^{1/p} |a_j|, \quad j = 1, 2, \dots, k,$$

Multiplying these inequalities for  $j = 1, 2, \dots, m$ , we obtain

$$\begin{aligned} \left( \prod_{j=0}^{p-1} |Q(\zeta_p^j)| \right)^{1/p} &= \left( \prod_{j=0}^m |a_j^p - 1| \right)^{1/p} \leq 2^{m/p} \left( \cos \frac{\eta}{2} \right)^{k/p} \prod_{j=0}^m \max\{|a_j|, 1\} \\ &\leq 2 \left( \cos \frac{\eta}{2} \right)^{k/p} M_0(Q). \end{aligned}$$

□

*Proof of Lemma 3.7.* By Lemma 3.1 If  $\zeta_p = e^{2i\pi/p}$  then, for all  $k$  not divisible by  $p$  we have  $|f_p(\zeta_p^k)| = \sqrt{p}$ , and hence  $|F_p(\zeta_p^k)| = \sqrt{p}$ . Moreover

$$F_p(\zeta_p^k) = (\zeta_p^k)^{-p/2} \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta_p^{ak} = (-1)^k \left(\frac{k}{p}\right) \sum_{a=1}^{p-1} \left(\frac{ak}{p}\right) \zeta_p^{ak} = (-1)^k \left(\frac{k}{p}\right) F_p(\zeta_p).$$

Therefore if  $\left(\frac{k}{p}\right) = \left(\frac{k+1}{p}\right)$ , then  $H_p\left(\frac{k}{p}\right)$  and  $H_p\left(\frac{k+1}{p}\right)$  have different signs. Since  $H_p(t)$  is real-valued and continuous on the real line, it must have a zero  $k/p$  and  $(k+1)/p$  by the Intermediate Value Theorem. However, by Lemma 2 in [C-00] we have

$$\left| \left\{ k \in \{1, 2, \dots, p-2\} : \left(\frac{k}{p}\right) = \left(\frac{k+1}{p}\right) \right\} \right| = \frac{p-3}{2},$$

and hence the values of  $k \in \{0, 1, \dots, p-1\}$  for which  $H_p$  has a zero between  $k/p$  and  $(k+1)/p$  is at least  $(p-3)/2$ .  $\square$

*Proof of Lemma 3.9.* Note that

$$I_\delta := \int_0^\infty \sin(\delta\pi x) C(x) \frac{dx}{x}$$

converges for every fixed  $\delta > 0$ , and

$$\lim_{\delta \rightarrow 0+} I_\delta = 0.$$

Indeed, there is an absolute constant  $c_1 > 0$  such that

$$C(x) \leq c_1 2^{-3x/\pi}, \quad x \geq 1,$$

as

$$\left| \cos\left(\frac{2x}{2k+1}\right) \right| < \frac{1}{2}, \quad \frac{3x}{\pi} < 2k+1 < \frac{6x}{\pi}.$$

Also,

$$\left| \frac{\sin(\delta\pi x)}{x} \right| \leq \delta\pi, \quad x > 0.$$

Therefore

$$I_\delta \leq \int_0^\infty \left| \frac{\sin(\delta\pi x)}{x} \right| |C(x)| dx \leq A_\delta + B_\delta,$$

where

$$A_\delta := \int_0^{\delta^{-1/2}} \left| \frac{\sin(\delta\pi x)}{x} \right| |C(x)| dx \leq \delta^{-1/2} \delta\pi \leq \delta^{1/2} \pi,$$

and

$$B_\delta := \int_{\delta^{-1/2}}^\infty \frac{|C(x)|}{x} dx \leq \delta^{1/2} \int_{\delta^{-1/2}}^\infty c_1 2^{-3x/\pi} dx \leq \delta^{1/2} \frac{c_1 \pi}{3 \log 2}.$$

So by choosing  $\delta > 0$  so that

$$I_\delta \leq A_\delta + B_\delta \leq \delta^{1/2}\pi + \delta^{1/2} \frac{c_1\pi}{3\log 2} \leq \frac{\pi\varepsilon}{2},$$

the lemma follows from Lemma 3.8.  $\square$

*Proof of Lemma 3.10.* Suppose there are  $0 \leq k_1 < k_2 < \dots < k_m \leq p-1$  such that

$$t_j \in I_{k_j}, \quad |f'_p(t_j)| \geq \gamma p^{3/2}, \quad j = 1, 2, \dots, m.$$

Then

$$0 < t_1 < t_2 < \dots < t_m < 2\pi,$$

and

$$\delta := \min \{t_2 - t_1, t_3 - t_2, \dots, t_m - t_{m-1}, 2\pi - (t_m - t_1)\} \geq \frac{\pi}{p}.$$

Hence by the large sieve inequality formulated in Lemma 3.6 and the Parseval formula applied to  $g_p(z) := z^{(3-p)/2} f'_p(z)$  we get

$$\begin{aligned} m\gamma^2 p^3 &\leq \sum_{j=1}^m |f'_p(e^{it_j})|^2 = \sum_{j=1}^m |g_p(e^{it_j})|^2 \\ &\leq \left( \frac{2(p-1)/2+1}{2\pi} + \delta^{-1} \right) \int_0^{2\pi} |g_p(e^{it})|^2 dt \\ &= \left( \frac{2(p-1)/2+1}{2\pi} + \delta^{-1} \right) \int_0^{2\pi} |f'_p(e^{it})|^2 dt \\ &\leq \left( \frac{p}{2\pi} + \frac{p}{\pi} \right) 2\pi \frac{(p-1)p(2p-1)}{6} = 3p \frac{(p-1)p(2p-1)}{6} \\ &\leq \frac{p^4}{2}. \end{aligned}$$

$\square$

*Proof of Lemma 3.11.* Let  $\varepsilon > 0$ . By Lemma 3.9 there is a  $\delta > 0$  depending only on  $\varepsilon > 0$  such that

$$(4.1) \quad \left| \left\{ k \in \{1, 2, \dots, p\} : |f_p(e^{i(2k+1)\pi/p})| > \delta\sqrt{p} \right\} \right| \geq (1 - \varepsilon/2)p$$

for all sufficiently large primes  $p \geq N_\varepsilon$ . Let  $\gamma := \varepsilon^{-1/2}$ . By Lemma 3.10 we have

$$(4.2) \quad \left| \left\{ k \in \{0, 1, \dots, p-1\} : \max_{z \in I_{k, \pi/2}} |f'_p(z)| \leq \gamma p^{3/2} \right\} \right| \geq p - \gamma^{-2}p/2 = (1 - \varepsilon/2)p.$$

Now let

$$A_{p, \delta, \gamma} := \left\{ k \in \{1, 2, \dots, p\} : |f_p(e^{i(2k+1)\pi/p})| > \delta\sqrt{p}, \quad \max_{z \in I_{k, \pi/2}} |f'_p(z)| \leq \gamma p^{3/2} \right\}.$$

By (4.1) and (4.2) we obtain

$$(4.3) \quad |A_{p,\delta,\gamma}| \geq (1 - \varepsilon)p.$$

Let  $0 < \eta < \min\{\delta/\gamma, \pi/2\}$ . Observe that  $k \in A_{p,\delta,\gamma}$  implies that  $f_p$  does not vanish in  $I_{k,\eta}$ . Indeed,  $z := e^{it} \in I_{k,\eta}$  implies

$$\left| t - \frac{(2k+1)\pi}{p} \right| < \frac{\eta}{p},$$

and hence

$$\begin{aligned} |f_p(z)| &\geq |f_p(e^{i(2k+1)\pi/p})| - |f_p(z) - f_p(e^{i(2k+1)\pi/p})| \\ &> \delta\sqrt{p} - \int_{(2k+1)\pi/p}^t f'_p(e^{i\tau}) e^{i\tau} d\tau \\ &\geq \delta\sqrt{p} - \int_{(2k+1)\pi/p}^t |f'_p(e^{i\tau})| |e^{i\tau}| d\tau \geq \delta\sqrt{p} - \frac{\eta}{p} \gamma p^{3/2} \\ &\geq \delta\sqrt{p} - \delta\sqrt{p} = 0 \end{aligned}$$

for all sufficiently large primes  $p \geq N_\varepsilon$ , and the lemma follows from (4.3).  $\square$

#### PROOF OF THEOREM 2.1

Now we are ready to prove the theorem.

*Proof of Theorem 2.1.* As in Lemma 3.11 let the subarcs  $I_{k,\eta}$  of the unit circle  $\partial D$  be defined by

$$I_{k,\eta} := \left\{ e^{it} : \left| t - \frac{(2k+1)\pi}{p} \right| < \frac{\eta}{p} \right\}, \quad k = 0, 1, \dots, p-1.$$

It follows from Lemma 3.11 that for  $\varepsilon := 1/8$  there is an  $\eta > 0$  such that

$$|\{k \in \{0, 1, \dots, p-1\} : f_p(z) \neq 0 \text{ for all } z \in I_{k,\eta}\}| \geq i \frac{7p}{8}$$

for all sufficiently large primes  $p$ . Combining this with Lemma 3.7 we have that

$$|\{k \in \{0, 1, \dots, p-1\} : f_p \text{ has a zero on } I_{k,\pi} \text{ and } f_p(z) \neq 0 \text{ for all } z \in I_{k,\eta}\}| \geq p/4$$

for all sufficiently large primes  $p$ . Hence the assumptions of Lemma 3.3 are satisfied with  $Q := f_p$  and  $k \geq p/4$  for all sufficiently large primes. Suppose that 1 is a zero of  $f_p$  with multiplicity  $m = m(p)$ . By either Lemma 3.4 or Lemma 3.5 we have  $m = O(p^{1/2})$ . Let  $g_p(z) := f_p(z)/h_m(z)$  with  $h_m(z) := (z-1)^m$ . Note that  $|g_p(1)|$  is a nonzero integer, hence  $|g_p(1)| \geq 1$ . Also,  $h_m$  is monic and has all its zeros on the unit circle, hence  $M_0(h_m) = 1$ .

Combining these with the multiplicative property of the Mahler measure, Lemma 3.3 applied to  $g_p$  with  $k \geq p/4$ , Lemma 3.1, and the fact that  $m = O(p^{1/2})$  implies that

$$\lim_{p \rightarrow \infty} p^{(-(1/2)+m)/p} = 1,$$

we conclude that there are absolute constants

$$c_1 := \left(2 \left(\cos \frac{\eta}{2}\right)^{1/4}\right)^{-1} > c_2 > \frac{1}{2}$$

such that

$$\begin{aligned} M_0(f_p) &= M_0(g_p)M_0(h_m) = M_0(g_p) \\ &\geq \left(2 \left(\cos \frac{\eta}{2}\right)^{k/p}\right)^{-1} \left(|g_p(1)| \prod_{k=1}^{p-1} |g_p(\zeta_p^k)|\right)^{1/p} \\ &\geq c_1 \left(|g_p(1)| \prod_{k=1}^{p-1} \left|\frac{f_p(\zeta_p^k)}{(\zeta_p^k - 1)^m}\right|\right)^{1/p} \geq c_1 \left(\prod_{k=1}^{p-1} \left|\frac{f_p(\zeta_p^k)}{(\zeta_p^k - 1)^m}\right|\right)^{1/p} \\ &= c_1 \frac{(p^{1/2})^{(p-1)/p}}{p^{m/p}} = c_1 p^{(p-1)/(2p)-m/p} = c_1 p^{1/2} p^{-((1/2)+m)/p} \\ &\geq c_2 \sqrt{p} \end{aligned}$$

for all sufficiently large primes  $p$ .  $\square$

**5. Acknowledgment.** The author thanks Stephen Choi for his careful reading of the paper and for his comments.

## REFERENCES

- B-90. R.C. Baker and H.L. Montgomery, *Oscillations of Quadratic L-functions*, in Analytic Number Theory (ed. B.C. Berndt et. al.), Birkhuser, Boston pp. 23–40, 1990.
- B-75. P.T. Bateman, G.B. Purdy, and S.S. Wagstaff, Jr., *Some numerical results on Fekete polynomials*, Math. Comp. (1975), 7–23.
- B-17. J. Bell and I. Shparlinski, *Power series approximations to Fekete polynomials*, J. Approx. Th. (to appear).
- B-73. E. Beller and D.J. Newman, *An extremal problem for the geometric mean of polynomials*, Proc. Amer. Math. Soc. **39** (1973), 313–317.
- B-02. P. Borwein, *Computational Excursions in Analysis and Number Theory*, Springer, CMS Books in Mathematics, Eds-in-Chief: Jonathan Borwein and Peter Borwein, New York, 2002.
- B-97a. P. Borwein and T. Erdélyi, *On the zeros of polynomials with restricted coefficients*, Illinois J. Math. (1997), 667–675.
- B-01a. P. Borwein and R. Lockhart, *The expected  $L_p$  norm of random polynomials*, Proc. Amer. Math. Soc. **129** (2001), 1463–1472.

- B-01b. P. Borwein and K.-K. S. Choi, *Merit factors of polynomials formed by Jacobi symbols*, Canad. J. Math. **53** (2001), no. 1, 33–50.
- B-01c. P. Borwein, K.-K. S. Choi, and S. Yazdani, *An extremal property of Fekete polynomials*, Proc. Amer. Math. Soc. **129** (2001), no. 1, 19–27.
- B-02. P. Borwein and K.-K. S. Choi, *Explicit merit factor formulae for Fekete and Turyn polynomials*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 129–234.
- B-04. P. Borwein, K.-K. S. Choi, and J. Jedwab, *Binary sequences with merit factor greater than 6.34*, IEEE Trans. Inform. Theory **50** (2004), no. 12, 3234–3249.
- B-97b. D. Boyd, *On a problem of Byrnes concerning polynomials with restricted coefficients*, Math. Comput. **66** (1997), 1697–1703.
- C-15. K.-K. S. Choi and T. Erdélyi, *On the average Mahler measures on Littlewood polynomials*, Proc. Amer. Math. Soc. Ser. B **1** (2015), 105–120.
- C-11. K.-K. S. Choi and M.J. Mossinghoff, *Average Mahler’s measure and  $L_p$  norms of unimodular polynomials*, Pacific J. Math. **252** (2011), no. 1, 31–50.
- C-35. S. Chowla, *Note on Dirichlet’s  $L$ -functions*, Acta Arith. **1** (1935), 113–114.
- C-00. B. Conrey, A. Granville, B. Poonen, and K. Soundararajan, *Zeros of Fekete polynomials*, Ann. Inst. Fourier (Grenoble) **50** (2000), 865–884.
- E-11. T. Erdélyi, *Sieve-type lower bounds for the Mahler measure of polynomials on subarcs*, Computational Methods and Function Theory **11** (2011), 213–228.
- E-07. T. Erdélyi and D. Lubinsky, *Large sieve inequalities via subharmonic methods and the Mahler measure of Fekete polynomials*, Canad. J. Math. **59** (2007), 730–741.
- E-12. T. Erdélyi, *Upper bounds for the  $L_q$  norm of Fekete polynomials on subarcs*, Acta Arith. **153** (2012), no. 1, 81–91.
- E-16. T. Erdélyi, *The Mahler measure of the Rudin-Shapiro polynomials*, Constr. Approx. **43** (2016), no. 3, 357–369.
- F-12. M. Fekete and G. Pólya, *Über ein Problem von Laguerre*, Rend. Circ. Mat. Palermo **34** (1912), 89–120.
- G-51. M.J. Golay, *Static multislit spectrometry and its application to the panoramic display of infrared spectra*, J. Opt. Soc. America **41** (1951), 468–472.
- G-16. C. Günther and K.-U. Schmidt,  *$L_q$  norms of Fekete and related polynomials*, manuscript.
- J-13a. J. Jedwab, D.J. Katz, and K.-U. Schmidt, *Advances in the merit factor problem for binary sequences.*, J. Combin. Theory Ser. A **120** (2013), no. 4, 882–906.
- J-13b. J. Jedwab, D.J. Katz, and K.-U. Schmidt, *Littlewood polynomials with small  $L_4$  norm*, Adv. Math. **241** (2013), 127–136.
- H-37. H. Heilbronn, *On real characters*, Acta Arith. **2** (1937), 212–213.
- H-82. L.K. Hua, *Introduction to Number Theory*, Springer-Verlag, New York-Berlin, 1982.
- J-16. J. Jung and S.W. Shin, *On the sparsity of positive-definite automorphic forms within a family*, J. Anal. Math. **129** (2016), no. 1, 105–138.
- L-66. J.E. Littlewood, *The real zeros and value distributions of real trigonometric polynomials*, J. London Math. Soc. **41** (1966), 336–342.

- L-87. D.S. Lubinsky, A. Máté, and P. Nevai, *Quadrature sums involving  $p$ th powers of polynomials*, SIAM J. Math. Anal. **18** (1987), 531–544.
- M-80. H.L. Montgomery, *An exponential polynomial formed with the Legendre symbol*, Acta Arith. **37** (1980), 375–380.
- M-78. H.L. Montgomery, *The analytic principle of the large sieve*, Bull. Amer. Math. Soc. **84** (1978), 547–567.
- P-19. G. Pólya, *Verschiedene Bemerkungen zur Zahlentheorie*, Jahresber. Dtsch. Math. Ver. **28** (1919), 31–40.
- S-51. H.S. Shapiro, *Master thesis*, MIT, 1951.

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843, COLLEGE STATION, TEXAS 77843 (T. ERDÉLYI)